Chapter 4

SYSTEMS OF LINEAR EQUATIONS

EXERCISE 4 (Page 174)

Solve the following systems of linear equations, the field of scalars being R:

1. $2x_1 + x_3 = 1$
$2x_1 + 4x_2 - x_3 = -2$
$x_1 - 8x_2 - 3x_3 = 2$

Sol. $A_b = \left[\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 2 & 4 & -1 & -2 \\ 1 & -8 & -3 & 2 \end{array}\right]$

$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -8 & -3 & 2 \\ 2 & 4 & -1 & -1 \\ 2 & 0 & 1 & 1 \end{array}\right]$, by R_{13}

$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -8 & -3 & 2 \\ 0 & 20 & 5 & -6 \\ 0 & 16 & 7 & -3 \end{array}\right]$, by $R_2 - 2R_1$ and $R_3 - 2R_1$

$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -8 & -3 & 2 \\ 0 & 4 & -2 & -3 \\ 0 & 16 & 7 & -3 \end{array}\right]$, by $R_2 - R_3$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -8 & -3 & 2 \\ 0 & 4 & -2 & -3 \\ 0 & 0 & 15 & 9 \end{array}\right], \quad \text{by } R_3 - 4R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & -7 & -4 \\ 0 & 4 & -2 & -3 \\ 0 & 0 & 1 & 3/5 \end{array}\right], \quad \text{by } R_1 + 2R_2 \text{ and } \frac{1}{15}R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/5 \\ 0 & 4 & 0 & -9/5 \\ 0 & 0 & 1 & 3/5 \end{array}\right], \quad \text{by } R_1 + 7R_3 \text{ and } R_2 + 2R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/5 \\ 0 & 1 & 0 & -9/20 \\ 0 & 0 & 1 & 3/5 \end{array}\right], \quad \text{by } \frac{1}{4}R_2$$

Hence $x_1 = \frac{1}{5}, \quad x_2 = -\frac{9}{20}, \quad x_3 = \frac{3}{5}$

2. $x_1 + x_2 + x_3 = a$
$x_1 + (1+a)x_2 + x_3 = 2a \quad , a \neq 0$
$x_1 + x_2 + (1+a)x_3 = 3a$

Sol. $A_b = \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 1+a & 1 & 2a \\ 1 & 1 & 1+a & 3a \end{array}\right]$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & a & 0 & a \\ 0 & 0 & a & 2a \end{array}\right], \quad \text{by } R_2 - R_1 \text{ and } R_3 - R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right], \quad \text{by } \frac{1}{a}R_2 \text{ and } \frac{1}{2a}R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 1 & a-1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right], \quad \text{by } R_1 - R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & a-3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right], \quad \text{by } R_1 - R_3$$

Therefore $x_1 = a - 3, \quad x_2 = 1, \quad x_3 = 2$

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3. $$\begin{aligned} x_1 - x_2 + x_3 - x_4 + x_5 &= 1 \\ 2x_1 + x_2 + 3x_3 + 4x_5 &= 3 \\ 3x_1 - 2x_2 + 2x_3 + x_4 + x_5 &= 1 \\ x_2 + x_4 + x_5 &= 0 \end{aligned}$$

Sol. $$A_b = \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 2 & 1 & 3 & 0 & 4 & 3 \\ 3 & -2 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 3 & 1 & 2 & 2 & 1 \\ 0 & 1 & -1 & 4 & -2 & -2 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right], \text{ by } R_2 - 2R_1 \text{ and } R_3 - 3R_1$$

$$\underset{\sim}{R} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 4 & -2 & -2 \\ 0 & 3 & 1 & 2 & 2 & 1 \end{array}\right], \text{ by } R_{24}$$

$$\underset{\sim}{R} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 3 & -3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right], \text{ by } R_1 + R_2, \; R_3 - R_2 \text{ and } R_4 - 3R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & -4 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right], \text{ by } R_1 - R_4 \text{ and } R_3 + R_4$$

$$\underset{\sim}{R} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 2 & -4 & -1 \end{array}\right], \text{ by } R_{34}$$

$$\underset{\sim}{R} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 & -1/2 \end{array}\right], \text{ by } \frac{1}{2}R_4$$

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$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 5 & 1/2 \\ 0 & 1 & 0 & 0 & 3 & 1/2 \\ 0 & 0 & 1 & 0 & -3 & 1/2 \\ 0 & 0 & 0 & 1 & -2 & -1/2 \end{array}\right], \quad \text{by } R_1 - R_4,\ R_2 - R_4 \text{ and } R_3 + R_4$$

We see that rank A = rank A_b = 4 < the number of unknowns (which are 5). Hence infinitely many solutions exist by assigning arbitrary value to any one of the unknowns, say x_5. Thus we have

$$x_1 + 5x_5 = \frac{1}{2} \Rightarrow x_1 = \frac{1}{2} - 5x_5$$

$$x_2 + 3x_5 = \frac{1}{2} \Rightarrow x_2 = \frac{1}{2} - 3x_5$$

$$x_3 - 3x_5 = \frac{1}{2} \Rightarrow x_3 = \frac{1}{2} + 3x_5$$

$$x_4 - 2x_5 = -\frac{1}{2} \Rightarrow x_4 = -\frac{1}{2} + 2x_5$$

4.
$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_2 + x_3 - x_4 &= 1 \\ x_3 + x_4 - x_5 &= 1 \\ -x_3 + x_4 + x_5 &= 1 \\ -x_2 + x_3 + x_4 &= 1 \end{aligned}$$

Sol.
$$A_b = \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{array}\right], \quad \text{by } R_1 - R_2 \text{ and } R_5 + R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & -2 & 2 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & -2 & 2 & 0 \end{array}\right], \quad \text{by } R_1 + 2R_3;\ R_2 - R_3;\ R_4 + R_3 \text{ and } R_5 - 2R_3$$

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$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & -2 & 2 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{array}\right], \quad \text{by } \frac{1}{2}R_4 \text{ and } \frac{1}{2}R_5$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right], \quad \text{by } R_1 - 3R_4,\ R_2 + 2R_4,\ R_3 - R_4 \text{ and } R_5 + R_4$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right], \quad \text{by } R_1 + 2R_5,\ R_2 - R_5, \text{ and } R_3 + R_5$$

Clearly rank A = rank $A_b = 5$ (the number of unknowns). Hence a unique solution exists and it is

$$x_1 = 1,\ x_2 = 1,\ x_3 = 1,\ x_4 = 1,\ x_5 = 1.$$

5.
$$\begin{aligned} x_1 - 2x_2 - 7x_3 + 7x_4 &= 5 \\ -x_1 + 2x_2 + 8x_3 - 5x_4 &= -7 \\ 3x_1 - 4x_2 - 17x_3 + 13x_4 &= 14 \\ 2x_1 - 2x_2 - 11x_3 + 8x_4 &= 7 \end{aligned}$$

Sol.
$$A_b = \left[\begin{array}{cccc|c} 1 & -2 & -7 & 7 & 5 \\ -1 & 2 & 8 & -5 & -7 \\ 3 & -4 & -17 & 13 & 14 \\ 2 & -2 & -11 & 8 & 7 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & -2 & -7 & 7 & 5 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 2 & 4 & -8 & -1 \\ 0 & 2 & 3 & -6 & -3 \end{array}\right], \quad \text{by } R_2 + R_1,\ R_3 - 3R_1 \text{ and } R_4 - 2R_1$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & 0 & -4 & 1 & 2 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 2 & 3 & -6 & -3 \end{array}\right], \quad \text{by } R_1 + R_4 \text{ and } R_3 - R_4$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & 0 & -4 & 1 & 2 \\ 0 & 2 & 3 & -6 & -3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & -2 & 2 \end{array}\right], \quad \text{by } R_{24}$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & 0 & -4 & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -3 & -\frac{3}{2} \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & -2 & 2 \end{array}\right], \quad \text{by } \frac{1}{2}R_2$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & 0 & -4 & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -3 & -\frac{3}{2} \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & -4 & 4 \end{array}\right], \quad \text{by } R_4 - R_3$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & 0 & -4 & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -3 & -\frac{3}{2} \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right], \quad \text{by } -\frac{1}{4}R_4$$

We note that rank A = rank A_b = 4 (the number of unknowns). Hence a unique solution exists and is given by

$$x_1 - 4x_3 + x_4 = 2 \qquad (1)$$

$$x_2 - \frac{3}{2}x_3 - 3x_4 = -\frac{3}{2} \qquad (2)$$

$$x_3 + 2x_4 = -2 \qquad (3)$$

$$x_4 = -1 \qquad (4)$$

(3) and (4) $\Rightarrow$ $x_3 - 2 = -2 \Rightarrow x_3 = 0$

(2) $\Rightarrow$ $x_2 - \frac{3}{2}(0) + 3 = -\frac{3}{2} \Rightarrow x_2 = -\frac{9}{2}$

(1) $\Rightarrow$ $x_1 - 4(0) - 1 = 2 \Rightarrow x_1 = 3$

6. $x_1 + 2x_2 + x_3 = -1$

$6x_1 + x_2 + x_3 = -4$

$2x_1 - 3x_2 - x_3 = 0$

$x_1 - x_2 = 1$

Sol. $A_b = \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 6 & 1 & 1 & -4 \\ 2 & -3 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right]$

$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 6 & 1 & 1 & -4 \\ 2 & -3 & -1 & 0 \\ 1 & 2 & 1 & -1 \end{array}\right]$, by R_{14}

$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 7 & 1 & -10 \\ 0 & -1 & -1 & -2 \\ 0 & 3 & 1 & -2 \end{array}\right]$, by $R_2 - 6R_1, R_3 - 2R_1$ and $R_4 - R_1$

$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 0 & -6 & -24 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & -2 & -8 \end{array}\right]$, by $R_1 - R_3, R_2 + 7R_3$ and $R_4 + 3R_3$

$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & -6 & -24 \\ 0 & 0 & -2 & -8 \end{array}\right]$, by R_{23}

$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right]$, by $-1R_2, -\frac{1}{6}R_3$ and $-\frac{1}{2}R_2$

$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$, by $R_4 - R_3$

$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$, by $R_1 - R_3$ and $R_2 - R_3$

Hence rank A = rank A_b = 3 (the number of unknowns). Therefore a unique solution exists and it is

$$x_1 = -1, \quad x_2 = -2, \quad x_3 = 4.$$

7. $\begin{aligned} 2x_1 + x_2 + 5x_3 &= 4 \\ 3x_1 - 2x_2 + 2x_3 &= 2 \\ 5x_1 - 8x_2 - 4x_3 &= 1 \end{aligned}$

Sol. $A_b = \left[\begin{array}{ccc|c} 2 & 1 & 5 & 4 \\ 3 & -2 & 2 & 2 \\ 5 & -8 & -4 & 1 \end{array}\right]$

$\underset{\sim}{R}\left[\begin{array}{ccc|c} 2 & 1 & 5 & 4 \\ 1 & -3 & -3 & -2 \\ 5 & -8 & -4 & 1 \end{array}\right]$, by $R_2 - R_1$

$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -3 & -3 & -2 \\ 2 & 1 & 5 & 4 \\ 5 & -8 & -4 & 1 \end{array}\right]$, by R_{12}

$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -3 & -3 & -2 \\ 0 & 7 & 11 & 8 \\ 0 & 7 & 11 & 11 \end{array}\right]$, by $R_2 - 2R_1$ and $R_3 - 5R_1$

$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -3 & -3 & -2 \\ 0 & 7 & 11 & 8 \\ 0 & 0 & 0 & 3 \end{array}\right]$, by $R_3 - R_2$

$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -3 & -3 & -2 \\ 0 & 1 & \frac{11}{7} & \frac{8}{7} \\ 0 & 0 & 0 & 1 \end{array}\right]$, by $\frac{1}{7}R_2$ and $\frac{1}{3}R_3$

Thus rank $A = 2$ and rank $A_b = 3$, so rank $A \neq$ rank A_b. hence solution of the system does not exist.

8. **Solve the system of equations having the given matrices as their augmented matrices:**

(i) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$

Sol. Let $A = \left[\begin{array}{ccc} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right]$, $A_b = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$

$A_b \underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$ by $R_2 - R_1$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right] \quad \text{by } R_3 + R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right] \quad \text{by } R_1 + 2R_2$$

Rank $A = 2$ and Rank $A_b = 3$. The system is inconsistent.

8.(ii) $\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 0 \end{array}\right]$

Sol. $A = \left[\begin{array}{ccc} 1 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{array}\right]$

$$\underset{\sim}{R}\left[\begin{array}{ccc} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 0 & 0 & -3 \\ 0 & 1 & -1 \end{array}\right] \quad \text{by } R_2 - R_1,\ R_3 - R_1 \text{ and } R_4 - R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 0 & 0 & -3 \\ 0 & 0 & -4 \end{array}\right] \quad \text{by } R_4 + R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{array}\right] \quad \text{by } R_4 - \frac{4}{3}R_3$$

Rank $A = 3$ = number of variables. The system has no nontrivial solution by (4.12).

8.(iii) $\left[\begin{array}{ccc|c} 4 & 2 & -1 & 0 \\ 3 & 3 & 6 & 3 \\ 5 & 1 & -8 & -1 \end{array}\right]$

Sol. $\left[\begin{array}{ccc|c} 4 & 2 & -1 & 0 \\ 3 & 3 & 6 & 3 \\ 5 & 1 & -8 & -1 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccc|c} 4 & 2 & -1 & 0 \\ 1 & 1 & 2 & 1 \\ 5 & 1 & -8 & -1 \end{array}\right]$ by $\frac{1}{3}R_2$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 4 & 2 & -1 & 0 \\ 5 & 1 & -8 & -1 \end{array}\right] \quad \text{by } R_{12}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -2 & -9 & -4 \\ 0 & -4 & -18 & -6 \end{array}\right] \quad \text{by } R_2 - 4R_1 \text{ and } R_3 - 5R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -2 & -9 & -4 \\ 0 & 0 & 0 & 2 \end{array}\right] \quad \text{by } R_3 - 2R_2$$

Rank $A = 2 \neq$ Rank $A_b = 3$. System inconsistent.

8.(iv) $A_b = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 0 & 4 & 1 & 8 \\ 0 & 0 & 2 & 0 \end{array}\right]$

Sol. The matrix is in echelon form. We reduce it to reduced echelon form.

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 2 & -1 & 0 & 4 \\ 0 & 4 & 0 & 8 \\ 0 & 0 & 2 & 0 \end{array}\right] \quad \text{by } R_2 - \frac{1}{2}R_3 \text{ and } R_1 - \frac{3}{2}R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 2 & 0 & 0 & 6 \\ 0 & 4 & 0 & 8 \\ 0 & 0 & 2 & 0 \end{array}\right] \quad \text{by } R_1 + \frac{1}{4}R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \text{by } \frac{1}{2}R_1, \frac{1}{4}R_2 \text{ and } \frac{1}{2}R_3$$

$x_1 = 3, \quad x_2 = 2, \quad x_3 = 0$

For what values of λ do the following homogeneous equations have nontrivial solutions? Find these solutions: (Problems 9 – 11):

9. $$\begin{aligned} (1-\lambda)x_1 + x_2 &= 0 \\ x_1 + (1-\lambda)x_2 &= 0 \end{aligned}$$

Sol. These homogenous equation will have nontrivial solutions if rank A < number of unknowns (= 2). This implies that the matrix A should be singular, where

$$A = \left[\begin{array}{cc} 1-\lambda & 1 \\ 1 & 1-\lambda \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{cc} 1 & 1-\lambda \\ 1-\lambda & 1 \end{array}\right], \quad \text{by } R_{12}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 1-\lambda \\ 1-\lambda-1 & 1-(1-\lambda) \end{bmatrix}, \quad \text{by } R_2 - R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 1-\lambda \\ -\lambda & \lambda \end{bmatrix}$$

For the nontrivial solution, rank $A < 2$ (the number of unknowns), so we must have $\lambda = 0$.

$$A \underset{\sim}{R}\begin{bmatrix} 1 & 1-\lambda \\ -\lambda & \lambda \end{bmatrix}$$

$$\Rightarrow \quad x_1 + x_2 = 0 \quad \text{or} \quad x_2 = -x_1$$

If $\lambda \neq 0$, then $A \underset{\sim}{R}\begin{bmatrix} 1 & 1-\lambda \\ -1 & 1 \end{bmatrix}$, by $\frac{1}{\lambda}R_2$.

$$\underset{\sim}{R}\begin{bmatrix} 1 & 1-\lambda \\ 0 & 2-\lambda \end{bmatrix}, \quad \text{by } R_2 + R_1$$

Thus A is row equivalent to a singular matrix if $\lambda = 2$. Hence

$$A \underset{\sim}{R}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow x_1 - x_2 = 0 \quad \text{or} \quad x_2 = x_1$$

10. $(3-\lambda)x_1 - x_2 + x_3 = 0$
$x_1 - (1-\lambda)x_2 + x_3 = 0$
$x_1 - x_2 + (1-\lambda)x_3 = 0$

Sol. These homogeneous equations will have nontrivial solutions if rank A < number of unknowns (= 3). This implies that the matrix A should be singular. Now, we have

$$A = \begin{bmatrix} 3-\lambda & -1 & 1 \\ 1 & -(1-\lambda) & 1 \\ 1 & -1 & 1-\lambda \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 1-\lambda \\ 1 & -1+\lambda & 1 \\ 3-\lambda & -1 & \end{bmatrix}, \quad \text{by } R_{13}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 1-\lambda \\ 0 & \lambda & \lambda \\ 0 & 2-\lambda & 1-(3-4\lambda+\lambda^2) \end{bmatrix}, \quad \begin{matrix} * \\ \end{matrix} \quad \begin{matrix} \text{by } R_2 - R_1 \\ \text{and } R_3 - (3-\lambda)R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 1-\lambda \\ 0 & 1 & 1 \\ 0 & 2-\lambda & -2+4\lambda-\lambda^2 \end{bmatrix}, \quad \text{by } \left(\frac{1}{\lambda}\right)R_2 \text{ if } \lambda \neq 0$$

$$\sim\begin{bmatrix} 1 & 0 & 2-\lambda \\ 0 & 1 & 1 \\ 0 & 0 & -4+5\lambda-\lambda^2 \end{bmatrix}, \quad ** \quad \text{by } R_3 - (2-\lambda)R_2$$

Now A is row equivalent to a singular matrix if

$$-4+5\lambda-\lambda^2 = 0$$

or $$\lambda^2-5\lambda+4 = 0$$

or $$(\lambda-4)(\lambda-1) = 0$$

i.e., $\lambda = 1,\ 4.$

Where $\lambda = 4$, we have

$$A \underset{\sim}{R} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $x_1 - 2x_3 = 0, \quad x_2 + x_3 = 0$

or $x_1 = 2x_2,\ x_3 = -x_2.$

Therefore, giving arbitrary values to x_3 we can obtain infinitely many solutions of the given system.

When $\lambda = 1$, then from step '**', we have

$$A \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $x_1 + x_3 = 0, \quad x_2 + x_3 = 0$

or $x_1 = -x_3,\ x_2 = -x_2$

$\Rightarrow$ $x_1 = x_2 = -x_3$

Therefore, assigning arbitrary values to x_3 there exist infinitely many solutions of the given system.

When $\lambda = 0$, then from step '*', we have

$$A \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & -2 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x_1 = 0, \quad x_2 - x_3 = 0$ or $x_2 = x_3.$

11. $$\begin{aligned} (1-\lambda)x_1 + x_2 - x_3 &= 0 \\ x_1 - \lambda x_2 - 2x_3 &= 0 \\ x_1 + 2x_2 - \lambda x_3 &= 0 \end{aligned}$$

Sol. $$A = \begin{bmatrix} 1-\lambda & 1 & -1 \\ 1 & -\lambda & -2 \\ 1 & 2 & -\lambda \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 2 & -\lambda \\ 1 & -\lambda & -2 \\ 1-\lambda & 1 & -1 \end{bmatrix}$$ by R_{13}

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -\lambda \\ 0 & -\lambda-2 & -2+\lambda \\ 0 & 1-2(1-\lambda) & -1+\lambda(1-\lambda) \end{bmatrix},$$ by $R_2 - R_1$ and $R_3 - (1-\lambda)R_1$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -\lambda \\ 0 & 1 & \dfrac{-2+\lambda}{-\lambda-2} \\ 0 & -1+2\lambda & -1+\lambda-\lambda^2 \end{bmatrix}, \text{ by } \left(\frac{1}{-\lambda-2}\right) R_2, \text{ if } \lambda \neq = -2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -\lambda \\ 0 & 1 & \dfrac{-2+\lambda}{-\lambda-2} \\ 0 & 0 & -1+\lambda-\lambda^2-(-1+2\lambda)\dfrac{-2+\lambda}{-\lambda-2} \end{bmatrix}^{*}, \quad \text{by } R_3-(-1+2\lambda)R_2$$

Hence A will be row – equivalent to a singular matrix in which case its rank should be less than the number of unknowns (i.e., the system of equations has nontrivial solution), if

$$-1+\lambda+\lambda^2-(-1+2\lambda)\frac{-2+\lambda}{-\lambda-2} = 0$$

$$\Rightarrow \quad (\lambda+2)(-1+\lambda-\lambda^2)+(-2+\lambda)(-1+2\lambda) = 0$$

$$\text{or} \quad -\lambda+\lambda^2-\lambda^3-2+2\lambda-2\lambda^2+2-5\lambda+2\lambda^2 = 0$$

$$\text{or} \quad -\lambda^3+\lambda^2-4\lambda = 0$$

$$\text{or} \quad \lambda(\lambda^2-\lambda+4) = 0$$

or $\lambda = 0$, whereas the roots of $\lambda^2 - \lambda = 4 = 0$ are imaginary.

For $\lambda = 0$, we have from step *

$$A \underset{\sim}{R}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1+2x_2 = 0, \quad x_2+x_3 = 0$$

$$\text{or} \quad x_1 = -2x_3, \quad x_2 = -x_3.$$

In each of the following, use Gauss-Jordan method to reduce the given system to reduced echelon form, indicating the operations performed and determine the solution if any: (Problems 12 – 19):

12.
$$6x_1-6x_2+6x_3 = 6$$
$$2x_1-4x_2-6x_3 = 12$$
$$10x_1-5x_2+5x_3 = 30$$

Sol.
$$A_b = \left[\begin{array}{ccc|c} 6 & -6 & 6 & 6 \\ 2 & -4 & -6 & 12 \\ 10 & -5 & 5 & 30 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & -2 & -3 & 6 \\ 2 & -1 & 1 & 6 \end{array}\right], \quad \text{by } \frac{1}{6}R_1, \ \frac{1}{2}R_2 \text{ and } \frac{1}{5}R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -1 & -4 & 5 \\ 0 & 1 & -1 & 4 \end{array}\right], \quad \text{by } R_2 - R_1 \text{ and } R_3 - 2R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 0 & -5 & 9 \\ 0 & 1 & -1 & 4 \end{array}\right], \quad \text{by } R_1 + R_3 \text{ and } R_2 + R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & -\frac{9}{5} \\ 0 & 0 & -1 & 4 \end{array}\right], \quad \text{by } -\frac{1}{5}R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & -\frac{9}{5} \\ 0 & 0 & 0 & \frac{11}{5} \end{array}\right], \quad \text{by } R_3 + R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & \frac{11}{5} \\ 0 & 0 & 1 & -\frac{9}{5} \end{array}\right], \quad \text{by } R_{12}$$

Hence rank A = rank A_b, so that the equations have a unique solution

$$x_1 = 5, \quad x_2 = \frac{11}{5}, \quad x_3 = \frac{-9}{5}$$

13. $5x_1 + 5x_2 - x_3 = 0$
$10x_1 + 5x_2 + 2x_3 = 0$
$5x_1 + 15x_2 - 9x_3 = 0$

Sol. The matrix of coefficients is

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} & & & \\ 0 & -5 & & \text{by} \quad \text{and } R_3 - R_1 \\ 0 & 10 & & \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 5 & 5 & -1 & \\ & & & \\ & & & \end{array}\right]$$

The rank of A is $2 < 3$ (the number of variables). The system has a nontrivial solution by **(4.12)**.

Assigning arbitrary value to x_3, we find:

From R_2: $-5x_2 + 4x_3 = 0$ or $x_2 = \frac{4}{5}x_3$

From R_1: $5x_1 + 5x_2 - x_3 = 0$

or $5x_1 = x_3 - 5x_2 = -3x_3$ or $x_1 = \frac{-3}{5}x_3$.

The required solution is

$$x_1 = \frac{-3}{5}x_3,\ x_2 = \frac{4}{5}x_3,\ x_3 \text{ arbitrary.}$$

14. $$\begin{aligned} 5x_1 - 2x_2 + x_3 &= 3 \\ 3x_1 + 2x_2 + 7x_3 &= 5 \\ x_1 + x_2 + 3x_3 &= 2 \end{aligned}$$

Sol. $$A_b = \left[\begin{array}{ccc|c} 5 & -2 & 1 & 3 \\ 3 & 2 & 7 & 5 \\ 1 & 1 & 3 & 2 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 3 & 2 & 7 & 5 \\ 5 & -2 & 1 & 3 \end{array}\right], \quad \text{by } R_{13}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -1 & -2 & -1 \\ 0 & -7 & -14 & -7 \end{array}\right], \quad \text{by } R_2 - 3R_1 \text{ and } R_3 - 5R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -7 & -14 & -7 \end{array}\right], \quad \text{by } R_{13} \text{ and } -R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \text{by } R_1 - R_2 \text{ and } R_3 + 7R_2$$

Thus rank A = rank $A_b = 2 <$ (the number of unknowns). Hence the system has infinite number of solutions satisfying the equations:

$$x_1 + x_3 = 1, \quad x_2 + 2x_3 = 1.$$

or $x_1 = 1 - x_3,\ x_2 = 1 - 2x_3,$ where x_3 is arbitrary.

15. $5x_1 - 2x_2 + x_3 = 2$
$3x_1 + 2x_2 + 7x_3 = 3$
$x_1 + x_2 + 3x_3 = 2$

Sol. $A_b = \left[\begin{array}{ccc|c} 5 & -2 & 1 & 2 \\ 3 & 2 & 7 & 3 \\ 1 & 1 & 3 & 2 \end{array}\right]$

$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 3 & 2 & 7 & 3 \\ 5 & -2 & 1 & 2 \end{array}\right]$, by R_{13}

$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -1 & -2 & -3 \\ 0 & -7 & -14 & -8 \end{array}\right]$, by $R_2 - 3R_1$ and $R_3 - 5R_1$

$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 13 \end{array}\right]$, by $R_3 - 7R_2$

$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$, by $\frac{1}{13}R_3$

This shows that rank $A = 2$, rank $A_b = 3$. Thus rank $A \neq$ rank A_b.
Hence the system has no solution.

16. $2x_1 - x_2 + 3x_3 = 3$
$3x_1 + x_2 - 5x_3 = 0$
$4x_1 - x_2 + x_3 = 3$

Sol. $A_b = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 3 & 1 & -5 & 0 \\ 4 & -1 & 1 & 3 \end{array}\right]$

$\underset{\sim}{R}\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 1 & 2 & -8 & -3 \\ 0 & 1 & -5 & -3 \end{array}\right]$, by $R_2 - R_1$ and $R_3 - 2R_1$

$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 2 & -8 & -3 \\ 2 & -1 & 3 & 3 \\ 0 & 1 & -5 & -3 \end{array}\right]$, by R_{12}

$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 2 & -8 & -3 \\ 0 & -5 & 19 & 9 \\ 0 & 1 & -5 & -3 \end{array}\right]$, by $R_2 - 2R_1$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 2 & -8 & -3 \\ 0 & 1 & -5 & -3 \\ 0 & -5 & 19 & 9 \end{array}\right], \quad \text{by } R_{23}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & -6 & -6 \end{array}\right], \quad \text{by } R_1 - 2R_3 \text{ and } R_3 + 5R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right], \quad \text{by } \left(\frac{-1}{6}\right)R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right], \quad \text{by } R_1 - 2R_3$$

We note that rank A = rank A_b = 3 (the number of unknowns). Hence the system has a unique solution

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 1.$$

17.
$$\begin{aligned} x_1 + 3x_2 + 5x_3 - 4x_4 &= 1 \\ x_1 + 2x_2 + x_3 - x_4 + x_5 &= -1 \\ x_1 - 2x_2 + 3x_3 + 2x_4 - x_5 &= 3 \\ x_1 + 5x_2 + 3x_3 + x_4 + x_5 &= -11 \\ x_1 + 3x_2 - x_3 + x_4 + 2x_5 &= -3. \end{aligned}$$

Sol.
$$A_b = \left[\begin{array}{ccccc|c} 1 & 3 & 5 & -4 & 0 & 1 \\ 1 & 2 & 1 & -1 & 1 & -1 \\ 1 & -2 & 3 & 2 & -1 & 3 \\ 1 & 5 & 3 & 1 & 1 & -11 \\ 1 & 3 & -1 & 1 & 2 & -3 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & -1 & -4 & 3 & 1 & -2 \\ 0 & -5 & -2 & 6 & -1 & 2 \\ 0 & 2 & -2 & 5 & 1 & -12 \\ 0 & 0 & -6 & 5 & 2 & -4 \end{array}\right], \quad \text{by subtracting } R_1 \text{ from the remaining rows}$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & -5 & -2 & 6 & -1 & 2 \\ 0 & 2 & -2 & 5 & 1 & -12 \\ 0 & 0 & -6 & 5 & 2 & -4 \end{array}\right], \quad \text{by } -1\,R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & -7 & 5 & 3 & -5 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 18 & -9 & -6 & 12 \\ 0 & 0 & -10 & 11 & 3 & -16 \\ 0 & 0 & -6 & 5 & 2 & -4 \end{array}\right], \text{ by } R_1 - 3R_2,\ R_3 + 5R_2 \text{ and } R_4 - 2R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & -7 & 5 & 3 & -5 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & -10 & 11 & 3 & -16 \\ 0 & 0 & -6 & 5 & 2 & -4 \end{array}\right], \text{ by } R_3 + 3R_5$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & -7 & 5 & 3 & -5 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -10 & 11 & 3 & -16 \\ 0 & 0 & -6 & 5 & 2 & -4 \end{array}\right], \text{ by } \frac{1}{6}R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & -7 & 0 & 3 & -5 \\ 0 & 1 & 4 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -10 & 0 & 3 & \\ 0 & 0 & -6 & 0 & 2 & -4 \end{array}\right], \quad R_1 - 5R_3, R_2 + 3R_3, R_4 - 11R_3 \text{ and } R_5 - 5R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc} 1 & 0 & -7 & 0 & 3 \\ 0 & 1 & 4 & 0 & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & -10 & & \\ 0 & 0 & 3 & & \end{array}\right], \quad \text{by } -\frac{1}{2}R_5$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & -7 & 0 & & \\ 0 & 1 & 4 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -10 \\ 0 & 0 & 3 & 0 & -1 & 2 \end{array}\right], \quad \text{by } R_4 + 3R_5$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & -7 & 0 & 3 & -5 \\ 0 & 1 & 4 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 3 & 0 & -1 & 2 \end{array}\right], \quad \text{by } -1\,R_4$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 3 & 65 \\ 0 & 1 & 0 & 0 & -1 & -38 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & -1 & -28 \end{array}\right], \quad \text{by } R_1 + 7R_4,\ R_2 - 4R_4 \text{ and } R_5 - 3R_4$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -19 \\ 0 & 1 & 0 & 0 & 0 & -10 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & -1 & -28 \end{array}\right], \quad \text{by } R_1 + 3R_5 \text{ and } R_2 - R_5$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -19 \\ 0 & 1 & 0 & 0 & 0 & -10 \\ 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 28 \end{array}\right], \quad \text{by } -1\,R_5$$

Thus rank A = rank $A_b = 5$ (the number of unknowns). Hence the system has a unique solution

$$x_1 = -19,\ x_2 = -10,\ x_3 = 10,\ x_4 = 0,\ x_5 = 28.$$

18. $3x_1 + 2x_2 + 4x_3 = 7$
$2x_1 + x_2 + x_3 = 4$
$x_1 + 3x_2 + 5x_3 = 3$

Sol. $$A_b = \left[\begin{array}{ccc|c} 3 & 2 & 4 & 7 \\ 2 & 1 & 1 & 4 \\ 1 & 3 & 5 & 3 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 3 & 5 & 3 \\ 2 & 1 & 1 & 4 \\ 3 & 2 & 4 & 7 \end{array}\right], \quad \text{by } R_{13}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 3 & 5 & 3 \\ 0 & -5 & -9 & -2 \\ 0 & -7 & -11 & -2 \end{array}\right], \quad \text{by } R_2 - 2R_1 \text{ and } R_3 - 3R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 3 & 5 & 3 \\ 0 & 1 & 9/5 & 2/5 \\ 0 & 1 & 11/7 & 2/7 \end{array}\right], \quad \text{by } -\frac{1}{5}R_2 \text{ and } -\frac{1}{7}R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & -2/5 & 9/5 \\ 0 & 1 & 9/5 & 2/5 \\ 0 & 0 & -8/35 & -4/35 \end{array}\right], \quad \text{by } R_1 - 3R_2 \text{ and } R_3 - R_2$$

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$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & -2/5 & 9/5 \\ 0 & 1 & 9/5 & 2/5 \\ 0 & 0 & 1 & 1/2 \end{array}\right], \quad \text{by } -\frac{35}{8}R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1. & 0 & -1/2 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \quad \text{by } R_1+\frac{2}{5}R_3 \text{ and } R_2-\frac{9}{5}R_3$$

The required solution is $x_1 = 2,\ x_2 = -\frac{1}{2},\ x_3 = \frac{1}{2}$.

19. $5x_1 + 4x_3 + 2x_4 = 3$
$x_1 - x_2 + 2x_3 + x_4 = 1$
$4x_1 + x_2 + 2x_3 = 1$
$x_1 + x_2 + x_3 + x_4 = 0.$

Sol. $A_b = \left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right]$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 2 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right], \quad \text{by } R_1 - R_3$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 2 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 5 & -6 & -8 & -7 \\ 0 & 2 & -1 & -1 & -2 \end{array}\right], \quad \text{by } R_2 - R_1,\ R_3 - 4R_1 \text{ and } R_4 - R_1$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -4 & -6 & -3 \\ 0 & 2 & -1 & -1 & -2 \end{array}\right], \quad \text{by } -1\ R_2 \text{ and } R_3 - 2R_4$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & 0 & -2 & -4 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -4 & -6 & -3 \\ 0 & 0 & 7 & 11 & 4 \end{array}\right], \quad \text{by } R_4 - 2R_3$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & 0 & -2 & -4 & -1 \\ 0 & 1 & -4 & -6 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & \frac{11}{7} & \frac{4}{7} \end{array}\right], \quad \text{by } R_{23} \text{ and } \frac{1}{7}R_4$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{6}{7} & \frac{1}{7} \\ 0 & 1 & 0 & \frac{2}{7} & -\frac{5}{7} \\ 0 & 0 & 1 & \frac{11}{7} & \frac{4}{7} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right], \quad \text{by } R_1 + 2R_4,\ R_2 + 4R_4 \text{ and } R_3$$

$$R\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad \text{by } R_1 + \frac{6}{7}R_4,\ R \ \frac{2}{7}R_4 \text{ and } R_3 - \frac{11}{7}R_4$$

Clearly rank A = rank A_b = 4 (the number of unknowns). Hence the system has a unique solution

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = -1, x_4 = 1$$

20. Show that the system

$$2x_1 - x_2 + 3x_3 = a$$
$$3x_1 + x_2 - 5x_3 = b$$
$$-5x_1 - 5x_2 + 21x_3 = c$$

is inconsistent if $c \neq 2a - 3b$

Sol. A $\left|\begin{array}{ccc|c} 2 & -1 & 3 & a \\ 3 & 1 & -5 & b \\ -5 & -5 & 21 & c \end{array}\right|$

$$R\left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & a/2 \\ 3 & 1 & -5 & b \\ -5 & -5 & 21 & c \end{array}\right] \quad \text{by } \frac{1}{2}R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -1/2 & 3/2 & a/2 \\ 0 & 5/2 & -19/2 & b - 3a/2 \\ 0 & -15/2 & 57/2 & c + 5a/2 \end{array}\right], \quad \text{by } R_2 - 3R_1 \text{ and } R_3 + 5R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{a}{2} \\ 0 & \frac{5}{2} & -\frac{19}{2} & b - \frac{3a}{3} \\ 0 & 0 & 0 & c + \frac{5a}{2} + 3b - \frac{9a}{2} \end{array}\right], \quad \text{by } R_3 + 3R_2$$

The given system of equations will be inconsistent if

rank A < rank A_b.

We note that rank $A = 2$ and rank A_b should be 3 in order that rank $A <$ rank A_b. Thus rank $A_b = 3$ if $c + \frac{5a}{2} + 3b - \frac{9a}{2} \neq 0$

or $c - 2a + 3b \neq 0$

or $c \neq 2a - b$.

21. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. Find a matrix B such that $AB = I_2$. Does this mean A is invertible? Explain.

Sol. Let $B = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}^T$ so that $AB = I_2$. Then

$$AB = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 & x_4 \\ x_2 & x_5 \\ x_3 & x_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or $$\begin{bmatrix} x_1 + x_2 + 3x_3 & x_4 + x_5 + 2x_6 \\ x_1 + 2x_2 + 3x_3 & x_4 + 2x_5 + 3x_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence we have

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ x_1 + 2x_2 + 3x_3 &= 0 \\ x_4 + x_5 + 2x_6 &= 0 \\ x_4 + 2x_5 + 3x_6 &= 1 \end{aligned}$$

Augmented matrix of this system of equation is

$$\left[\begin{array}{cccccc|c} 1 & 1 & 2 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{cccccc|c} 1 & 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right], \text{ by } R_2 - R_1 \text{ and } R_4 - R_3$$

$$\underset{\sim}{R}\left[\begin{array}{cccccc|c} 1 & 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right], \text{ by } R_3 - R_4$$

From R_4: $x_5 + x_6 = 1$

or $x_5 = 1 - x_6$

From R_3: $x_4 + x_6 = -1$

or $x_4 = -1 - x_6$

From R_2: $x_2 + x_3 = -1$

or $x_2 = -1 - x_3$

From R_1: $x_1 + x_2 + 2x_3 = 1$

or $x_1 = 1 - x_2 - 2x_3$

$= 1 + 1 + x_3 - 2x_3$

$= 2 - x_3$

Let $x_3 = s$, $x_6 = t$, then

$x_1 = 2 - s$

$x_2 = -1 - s$

$x_4 = -1 - t$

$x_5 = 1 - t$

$$B = \begin{bmatrix} 2-s & -1-t \\ -1-s & 1-t \\ s & t \end{bmatrix}, \quad s, t \in R.$$

Verification:

$$AB = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2-s & -1-t \\ -1-s & 1-t \\ s & t \end{bmatrix}$$

$$= \begin{bmatrix} 2-s-1-s+2s & -1-t+1-t+2t \\ 2-s-2-2s+3s & -1-t+2-2t+3t \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

A is not invertible for the following reasons:

(i) B is not unique.

(ii) $AB = I_2$. But it can be easily checked that $BA \neq I_2$.

22. A soap manufacturer decides to spend 600,000 rupees on radio, magazine and TV advertising. If he spends as much on TV advertising as on magazines and radio together, and the amount spent on magazines and TV combined equals five times that spent on radio, what is the amount to be spent on each type of advertising?

Sol. Let x, y, z be the amounts in rupees spent on radio, magazines and TV advertising respectively. Then according to the given conditions, we have

$$x + y + z = 600{,}000 \qquad (1)$$

$$z = x + y$$

$$\Rightarrow \quad x + y - z = 0 \qquad (2)$$

$$y + z = 5x$$

$$\Rightarrow \quad 5x - y - z = 0 \qquad (3)$$

To solve the equations (1), (2) and (3), we use Gauss-Jordan method. We have

$$A_b = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 600{,}000 \\ 1 & 1 & -1 & 0 \\ 5 & -1 & -1 & 0 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 0 & 0 & 2 & 600{,}000 \\ 1 & 1 & -1 & 0 \\ 0 & -6 & 4 & 0 \end{array}\right], \quad \text{by } R_1 - R_2 \text{ and } R_3 - 5R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 0 & 0 & 2 & 600{,}000 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & -2/3 & 0 \end{array}\right], \quad \text{by } -\frac{1}{6}R_3$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 300{,}000 \\ 1 & 0 & -1/3 & 0 \\ 0 & 1 & 2/3 & 0 \end{array}\right], \quad \text{by } R_2 - R_3 \text{ and } \frac{1}{2}R_1$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 300{,}000 \\ 1 & 0 & 0 & 100{,}000 \\ 0 & 1 & 0 & 200{,}000 \end{array}\right], \quad \text{by } R_2 + \frac{1}{3}R_1 \text{ and } R_3 + \frac{2}{3}R_1$$

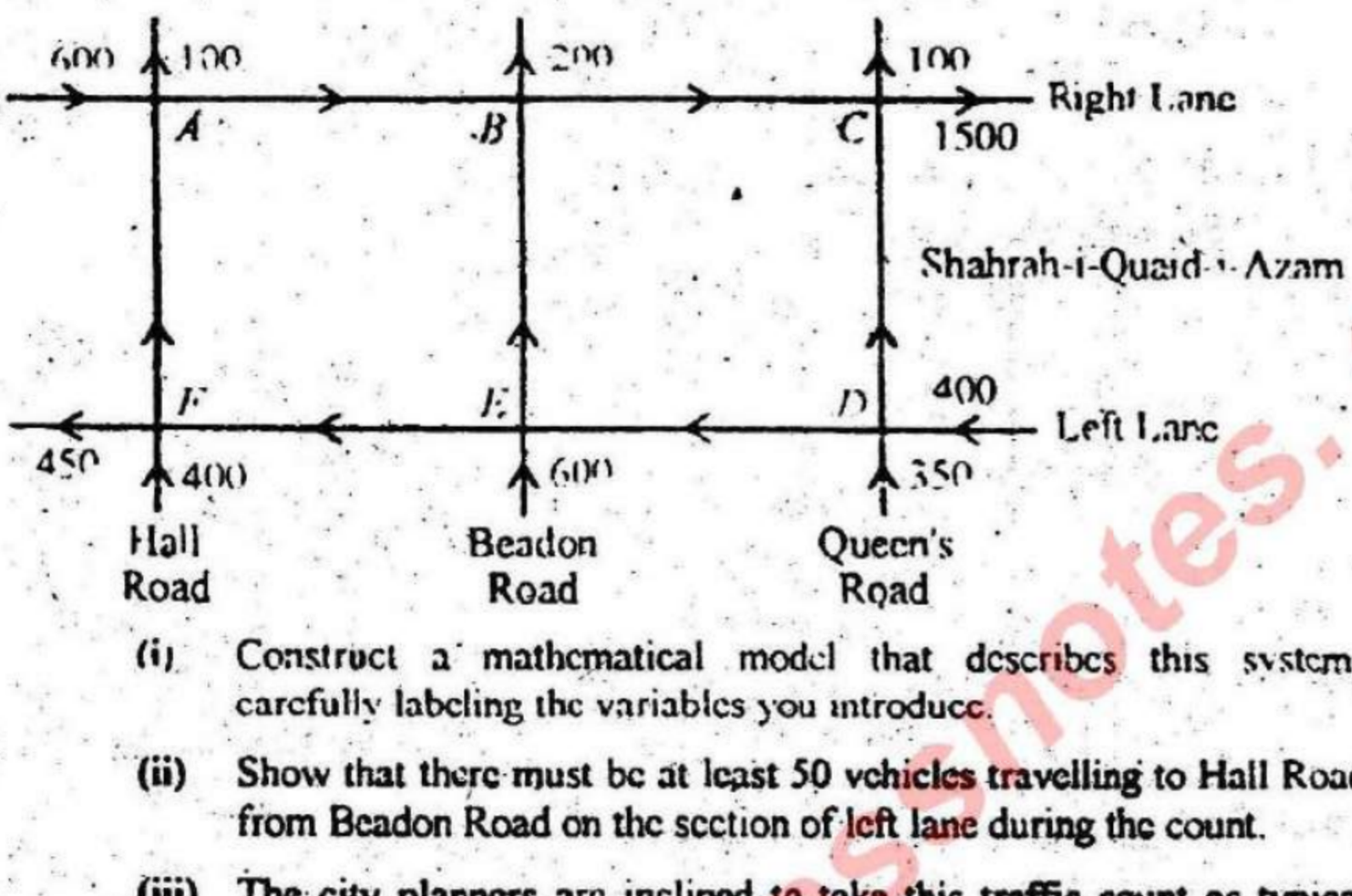
$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 100{,}000 \\ 0 & 0 & 1 & 200{,}000 \\ 0 & 1 & 0 & 300{,}000 \end{array}\right], \quad \text{by } R_{12}$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 100{,}000 \\ 0 & 1 & 0 & 200{,}000 \\ 0 & 0 & 1 & 300{,}000 \end{array}\right], \quad \text{by } R_{23}$$

Thus rank A = rank A_b = 3 (the number of unknowns). Hence the system has a unique solution

$$x = 10{,}000, \quad y = 200{,}000 \quad z = 300{,}000.$$

23. Traffic counters submitted the following information for March 23 from 7 p.m. to 8 p.m. on the following roads of the city

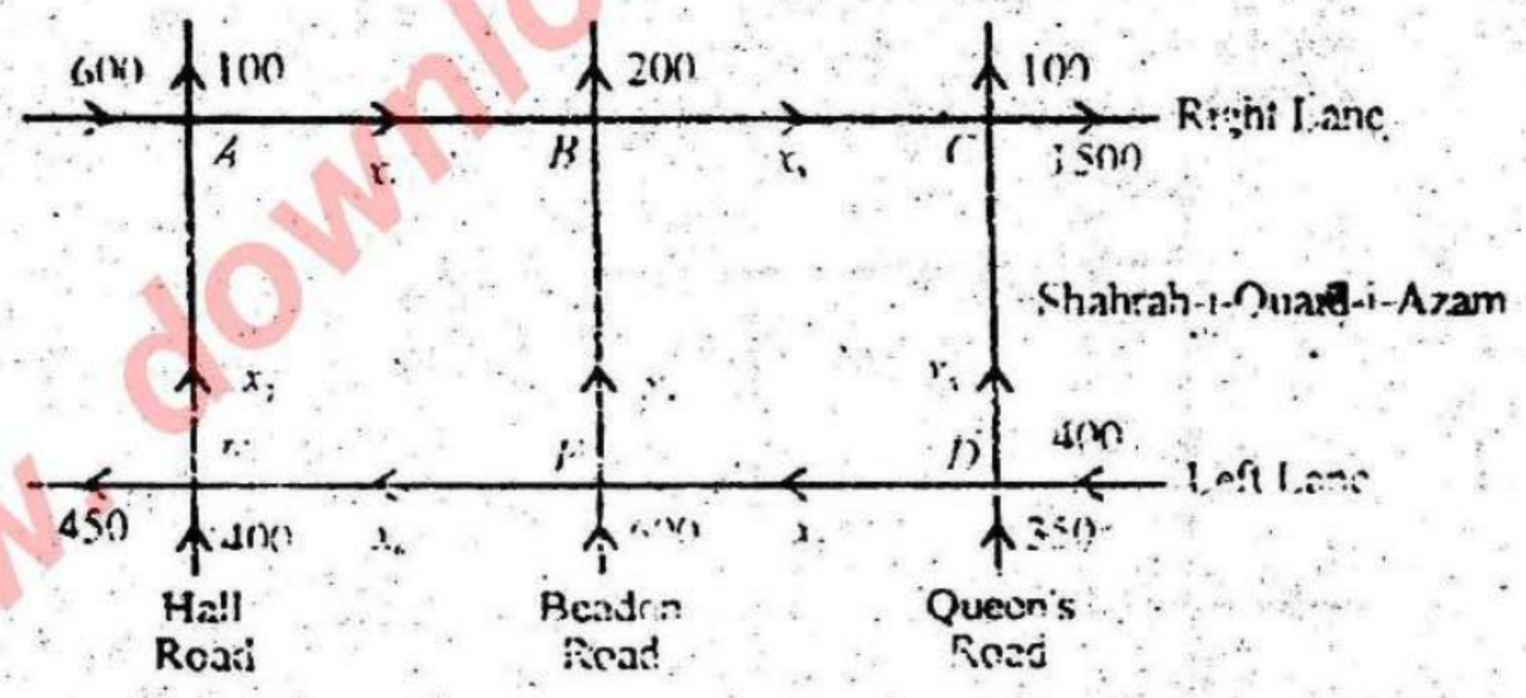


(i) Construct a mathematical model that describes this system, carefully labeling the variables you introduce.

(ii) Show that there must be at least 50 vehicles travelling to Hall Road from Beadon Road on the section of left lane during the count.

(iii) The city planners are inclined to take this traffic count as typical rush hour evening traffic in this area. In their planning of the annual closure of left lane between Queen's Road and Beadon Road for repair, how much traffic can be expected on right lane between Queen's Road and Beadon Road?

Sol.(i) Let x_1, x_2, x_3, x_4, x_5 denote the number of vehicles along different sections of various roads, as shown in the traffic chart



The required mathematical model can be described by the following equations.

The incoming traffic at A is x_2 and 600 vehicles and balance the sum of 100 and x_1. Thus

$$x_2 + 600 = x_1 + 100 \quad \Rightarrow \quad x_1 - x_2 = 500 \qquad (1)$$

Similarly at B: $x_1 + x_4 = x_3 + 200 \Rightarrow x_1 - x_3 + x_4 = 200$ (2)

At C: $x_3 + x_5 = 1500 + 100 \Rightarrow x_3 + x_5 = 1600$ (3)

At D: $x_6 + 400 = x_2 + 450 \Rightarrow x_2 - x_6 = -50$ (4)

At E: $x_7 + 600 = x_4 + x_6 \Rightarrow x_4 + x_6 - x_7 = 600$ (5)

At F: $x_5 + x_7 = 400 + 350 \Rightarrow x_6 + x_7 = 750$ (6)

From the above six equations, the augmented matrix of the system is

$$A_b = \left[\begin{array}{ccccccc|c} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 500 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1600 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -50 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 600 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 750 \end{array}\right]$$

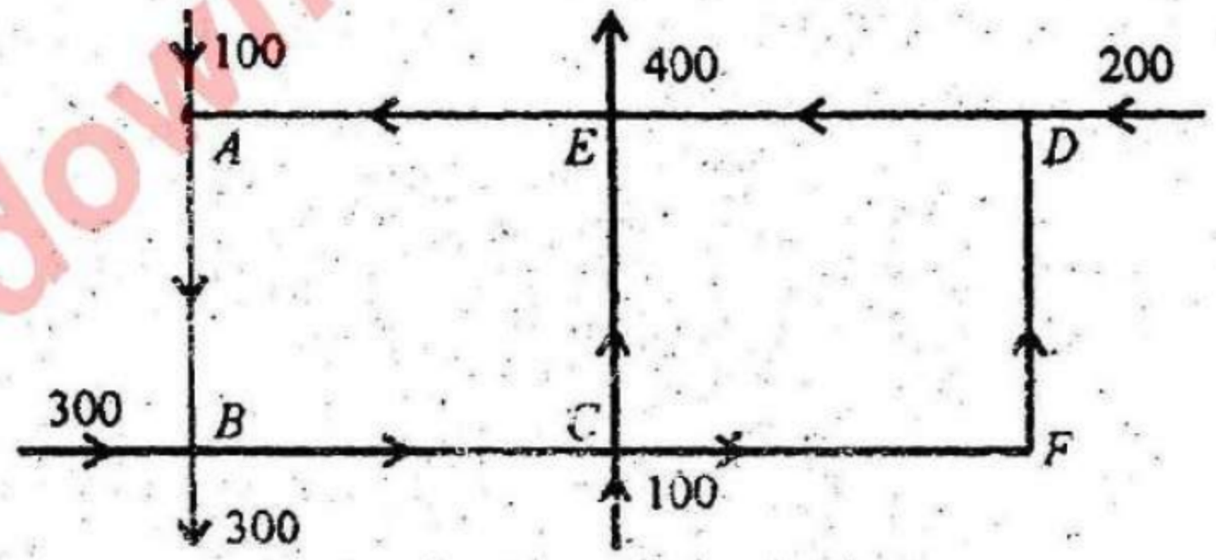
(ii) Equation (4) implies that $x_6 = x_2 + 50$. This shows that if $x_2 = 0$, then least number of vehicles traveling on the section of the left lane to Hall Road from Beadon Road during the count is 50.

(iii) On account of the closure of left lane between Queen's and Beadon for repair, we must have $x_7 = 0$. Then x_3, the number of vehicles expected on right lane between Queen's Road and on Beadon Road can be obtained from equations (3) and (6). Equation (6) implies $x_5 = 750$, and so substituting this into equation (3), we get

$$x_3 + 750 = 1600$$

$$\Rightarrow \quad x_3 = 1600 - 750 = 850.$$

24. One part of Lahore's network of traffic is given with the number of vehicles that enter and leave during a typical rush hour as shown below. All the lanes are one-way in the direction indicated by the arrows.

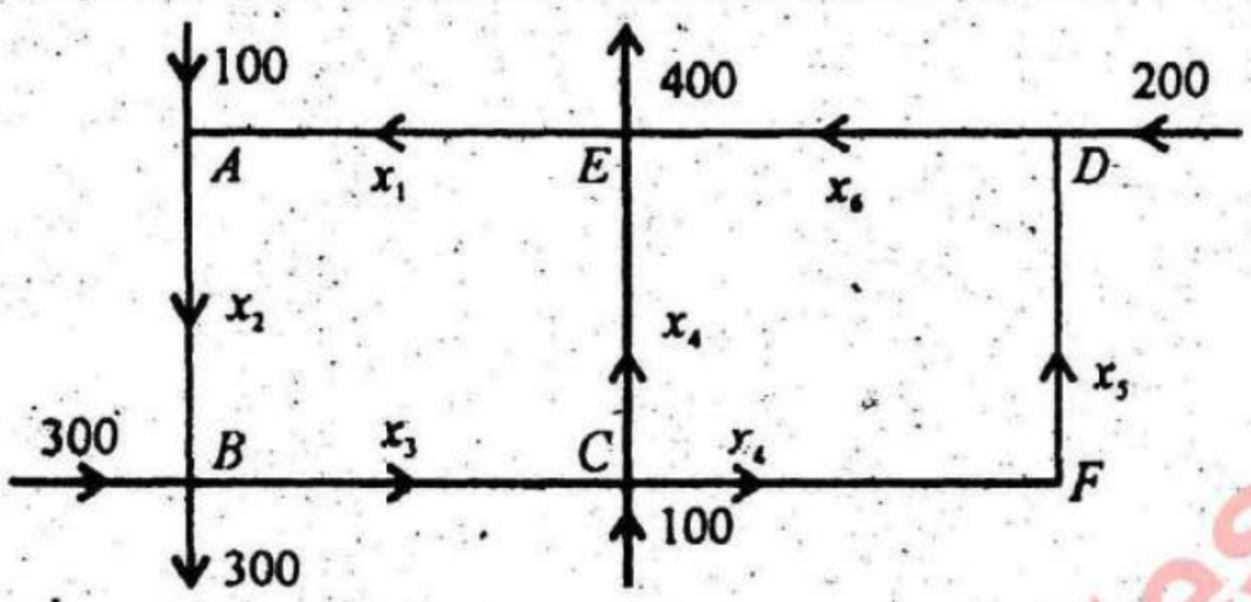


(i) Construct the linear mathematical model that describes this system.

(ii) If the stretch EA is closed for repair, what will be the traffic flow along the other stretches?

(iii) If only 100 vehicles are allowed to pass during the rush hour through EA, how will that affect the other branches?

Sol.(i) Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ denote the number of vehicles along different sections of various roads in the rush hour as shown below.



The required mathematical model can be described by the following equations.

The traffic flow entering at A is x_1 and 100 whose sum is equal to x_2 vehicles.

Thus at A: $x_1 + 100 = x_2$ (1)

Similarly at B: $x_2 + 300 = 300 + x_3$ (2)

At C: $x_3 + 100 = x_4 + x_5$ (3)

At D: $x_5 + 200 = x_6$ (4)

At E: $x_6 + x_4 = 400 + x_1$ (5)

(ii) The augmented matrix of this system of equations is

$$A_b = \left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & -100 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & -100 \\ 0 & 0 & 0 & 0 & 1 & -1 & -200 \\ 1 & 0 & 0 & -1 & 0 & -1 & -400 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & -100 \\ 0 & 1 & 0 & -1 & -1 & 0 & -100 \\ 0 & 0 & 1 & -1 & -1 & 0 & -100 \\ 0 & 0 & 0 & 0 & 1 & -1 & -200 \\ 1 & 0 & 0 & -1 & 0 & -1 & -400 \end{array}\right], \text{ by } R_2 + R_3$$

$$\underset{\sim}{R}\left[\begin{array}{cccccc|c} 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & -100 \\ 0 & 0 & 1 & -1 & -1 & 0 & -100 \\ 0 & 0 & 0 & 0 & 1 & -1 & -200 \\ 1 & 0 & 0 & -1 & 0 & -1 & -400 \end{array}\right], \quad \text{by } R_1 - R_2$$

$$R\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & -100 \\ 0 & 0 & 1 & -1 & -1 & 0 & -100 \\ 0 & 0 & 0 & 0 & 1 & -1 & -200 \\ 0 & 0 & 0 & 0 & 1 & -1 & -400 \end{bmatrix} \quad \text{by } R_5 - R_1$$

$$R\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & -100 \\ 0 & 0 & 1 & -1 & -1 & 0 & -100 \\ 0 & 0 & 0 & 0 & 1 & -1 & -200 \\ 0 & 0 & 0 & 0 & 0 & 0 & -200 \end{bmatrix} \quad \text{by } R_5 + R_4$$

Thus x_6 is arbitrary.

Let $x_6 = s$

When EA is closed, $x_1 = 0$. From equations (2) and (3) we have

$$x_2 = x_3 = 100.$$

From (4) and (5), we get

$$x_5 = x_6 - 200 = s - 200$$

$$x_4 = 400 - s$$

$$s \geq 200 \text{ and } 200 \leq s < 400.$$

The solution is not unique.

(iii) In this case $x_1 = 100$.

Equations (1) and (2) imply that $x_2 = 200$ and $x_3 = 200$

Eq. (5) implies. $x_4 + x_6 = 500$

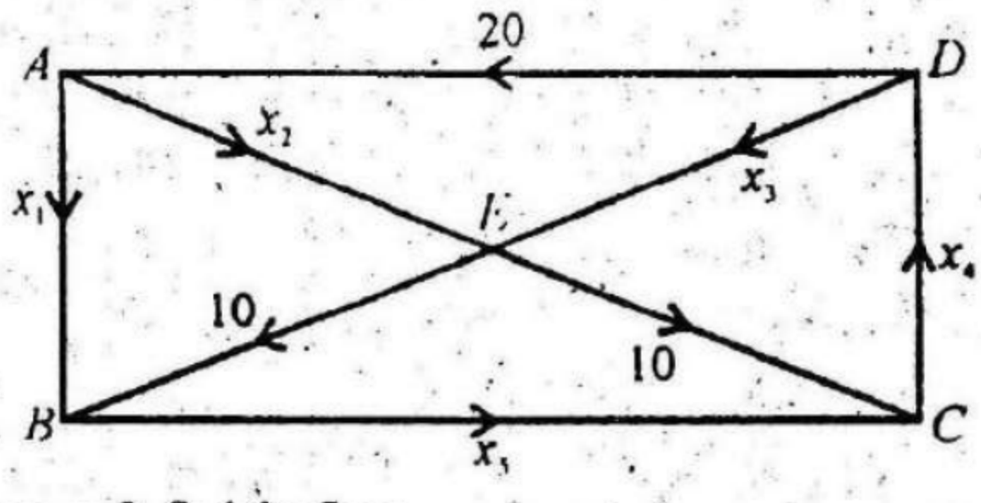
$\Rightarrow$ $x_4 + s = 500 \Rightarrow x_4 = 500 - s,\ 0 \leq s \leq 500$

Eq. (4) implies, $x_5 + 200 = s$ i.e., $x_5 = s - 200,\ 0 \leq s \leq 200$

Thus, $0 \leq s \leq 500$ and $0 \leq s \leq 200 \Rightarrow 0 \leq s \leq 200$

The solution is not unique.

5. Set up a system of linear equations to represent the network shown in the diagram and solve the system



If $x_5 = x_1 = 0$, find the flow

Sol. Equation of the flow at various junction arc as under.

At A $\quad 20 = x_1 + x_2$ (1)

B $\quad x_1 + 10 = x_3$ (2)

C $\quad x_3 + 10 = x_4$ (3)

D $\quad x_4 = x_5 + 20$ (4)

E $\quad x_2 + x_3 = 10 + 10$ (5)

We rewrite and rearrange these equation as

$$x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 = 20 \quad (1)$$

$$0x_1 + x_2 + x_3 + 0x_4 + 0x_5 = 20 \quad (2)$$

$$0x_1 + 0x_2 - x_3 + x_4 + 0x_5 = 20 \quad (3)$$

$$-x_1 + 0x_2 + 0x_3 + 0x_4 + x_5 = 10 \quad (4)$$

$$0x_1 + x_2 + 0x_3 + x_4 + 0x_5 = 10 \quad (5)$$

The augmented matrix of the system is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ -1 & 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{bmatrix}, \quad \text{by } R_4 + R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{bmatrix}, \quad \text{by } R_2 + R_4$$

Therefore, $x_4 = 10 + x_5 = 10 + t$, where $x_5 = t$

$x_2 = 30 - x_5 = 30 - t$

$-x_3 + x_4 = 20$

or $\quad x_3 = x_4 - 20 = 10 + t - 20 = t - 10$

$x_1 + x_2 = 20$

or $\quad x_1 = 20 - x_2 = 20 - (30 - t) = t - 10$

Thus

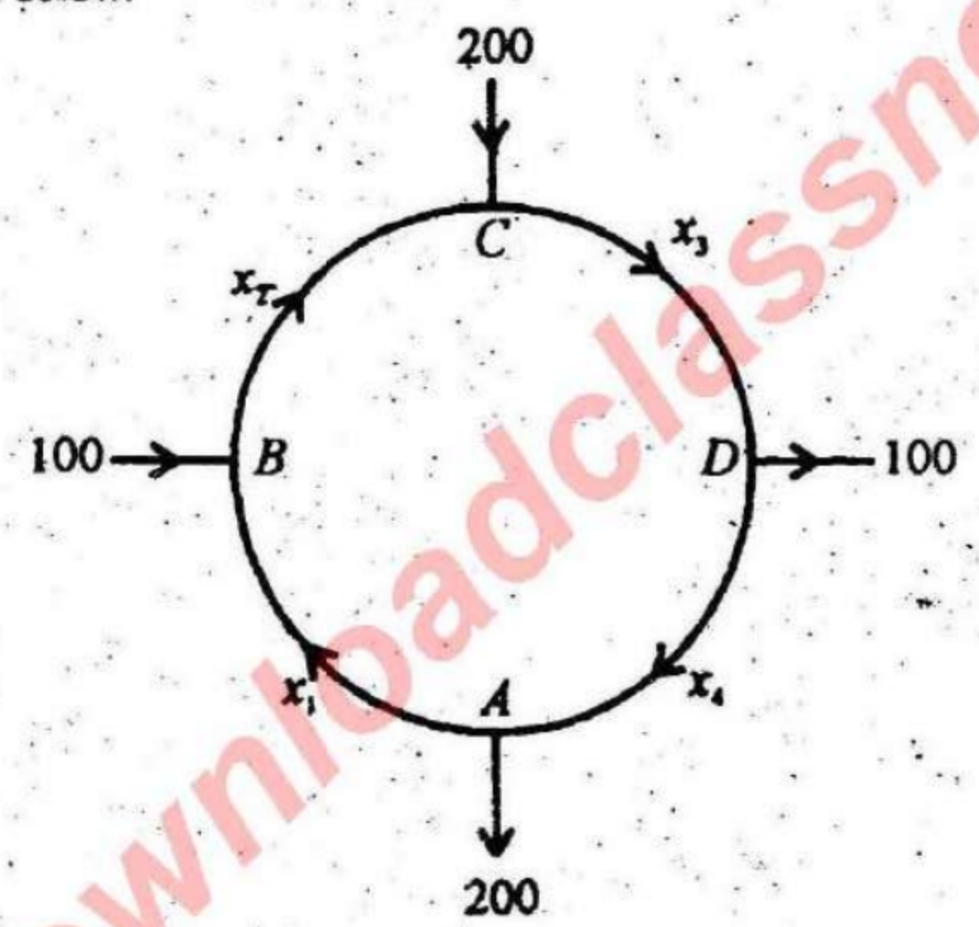
$$x_1 = t - 10$$
$$x_2 = 30 - t$$
$$x_3 = t - 10$$
$$x_4 = t + 10$$
$$x_5 = t \quad \text{being arbitrary.}$$

If $x_1 = x_3 = 0$,

then $t = 20$ **and so**

$$x_2 = 30 - 10 = 20$$
$$x_4 = 10 + 10 = 20$$
$$x_5 = 10.$$

26. **The flow of traffic at the Kalama Chowk on Ferozepur Road, Lahore i: shown below:**



(a) Solve the system.

(b) Find the traffic flow when $x_4 = 300$.

Sol. **Equations of the traffic flow at various points are as below:**

At A: $x_4 = 200 + x_1$ (1)

B: $x_1 + 100 = x_2$ (2)

C: $x_2 + 200 = x_3$ (3)

D: $x_3 + 100 = x_4$ (4)

Rewriting these equations, we have

$$-x_1 + 0x_2 + 0x_3 + x_4 = 200 \quad (1)$$
$$-x_1 + x_2 + 0x_3 + 0x_4 = 100 \quad (2)$$
$$0x_1 - x_2 + x_3 + 0x_4 = 200 \quad (3)$$
$$0x_1 + 0x_2 - x_3 + x_4 = 100 \quad (4)$$

The augmented matrix of the system is

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 200 \\ -1 & 1 & 0 & 0 & 100 \\ 0 & -1 & 1 & 0 & 200 \\ 0 & 0 & -1 & 1 & 100 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} -1 & 0 & 0 & 1 & 20 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & -1 & 1 & 0 & 200 \\ 0 & 0 & -1 & 1 & 100 \end{bmatrix}, \quad \text{by } R_2 - R_1$$

$$\underset{\sim}{R} \begin{bmatrix} -1 & 0 & 0 & 1 & 200 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & -1 & 1 & 100 \end{bmatrix}, \quad \text{by } R_3 + R_2$$

$$\underset{\sim}{R} \begin{bmatrix} -1 & 0 & 0 & 1 & 200 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 200 \end{bmatrix}, \quad \text{by } R_4 + R_3.$$

Therefore, $x_4 = t$, where t is arbitrary.

$$x_3 = 100 + x_4 = 100 + t$$

$$x_2 - x_4 = -100$$

or $$x_2 = t - 100$$

$$-x_1 + x_4 = 200$$

or $$x_1 = t - 200.$$

The solution is

$$x_1 = t - 200$$
$$x_2 = t - 100$$
$$x_3 = t - 100$$
$$x_4 = t.$$

When $x_4 = 300 = t$, then

$x_1 = 100$, $x_2 = 200$, $x_3 = 400$, $x_4 = 300$.